A NOTE ON THE FOURIER TRANSFORMS OF GENERALIZED FUNCTIONS ASSOCIATED WITH QUADRATICS FORMS

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RESUMEN: En este breve trabajo se demuestra un teorema, el Teorema 1, que permite evaluar la transformada de Fourier de distribuciones asociadas a formas cuadráticas, en particular los casos en los que tales distribuciones dependen de $\left(P+i\varepsilon|t|^2\right)$ donde *P* es una forma cuadrática no degenerada en n variables de la forma $P=t_1^2+\ldots+t_{\mathrm{p}}^2-t_{\mathrm{p+1}}^2-\ldots-t_{\mathrm{p+q}}^2$.
El teorema demostrado es análogo a otro de Trione que permite evaluar la transformada de Fourier de familias de funciones generalizadas dependientes de las distribuciones $(P+i0)$.
El resultado presentado resuelve situaciones en las que el teorema de Trione no parece aplicable de modo directo.

ABSTRACT: In this short paper a theorem that permits the evaluation of the Fourier transform of generalized functions associated with quadratics forms is obtained. Particularly cases such as distributions depending on $\left(P+i\varepsilon|t|^2\right)^\lambda$ where P is a non degenerate form in *n* variables $P=t_1^2+\cdots+t_p^2-t_{p+1}^2-\cdots-t_{p+q}^2$. The theorem proved is an analogue of theorem by Trione that allows us to evaluate the Fourier transform of a family of generalized functions depending on $(P+i0)$ distributions.

Palabras claves: Transformada de Fourier. Funciones generalizadas. Formas cuadráticas

Key words: Fourier transform. Generalized functions. Quadratics form.

I. INTRODUCTION

In this note we will consider distributional functions associated with the quadratic form P (t) defined as follows:

$$\mathrm{P}(t) = P(t) + i\varepsilon|t|^2, \qquad \text{(I,1)}$$

where P (t) is the non degenerate form in *n* variables

$$P=t_1^2+\cdots+t_p^2-t_{p+1}^2-\cdots-t_{p+q}^2 \qquad \text{(I,2)}$$

where $p+q=n$; *n* = dimension of the space, $|t|^2=t_1^2+\cdots+t_n^2$, and ε a real positive number.

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From the previous considerations it follow that P (t) can be written in the form

$$P(t) = (1+i\varepsilon)t_1^2 + \cdots + (1+i\varepsilon)t_p^2 + (-1+i\varepsilon)t_{p+1}^2 + \cdots + (-1+i\varepsilon)t_{p+q}^2 \qquad (I,3)$$

where all the coefficients abide the constraint that their imaginary part is positive. Gelfand [cf[1], p. 284] has evaluated the Fourier transform of generalized functions like P^{λ}, where P is a quadratic form with complex coefficients whose imaginary part is positive.

Therefore, according to Gelfand we put

$$F[P(t)^{\lambda}] = \frac{e^{-\frac{1}{4}\pi ni}\, 2^{2\lambda+n}\, \pi^{\frac{n}{2}}\, \Gamma\left(\lambda+\frac{n}{2}\right)}{\sqrt{-i\alpha_1}\cdots\sqrt{-i\alpha_n}\,\Gamma(-\lambda)}\left(\frac{s_1^2}{\alpha_1}+\cdots+\frac{s_n^2}{\alpha_n}\right)^{-\lambda-\frac{n}{2}} \qquad (I,4)$$

where λ is a complex number, and

$$\alpha_1 = \alpha_2 = \cdots = \alpha_p = (1+i\varepsilon),$$

$$\alpha_{p+1} = \alpha_{p+2} = \cdots = \alpha_{p+q} = (1+i\varepsilon),$$

and F denotes the Fourier transform.

After elementary calculations the following expression is obtained:

$$F[P(t)^{\lambda}] = \frac{e^{-\frac{1}{4}\pi ni}\, 2^{2\lambda+n}\, \pi^{\frac{n}{2}}\, \Gamma\left(\lambda+\frac{n}{2}\right)}{\left(\sqrt{\varepsilon-i}\right)^p\left(\sqrt{\varepsilon+i}\right)^q\,\Gamma(-\lambda)}\cdot\frac{1}{1+\varepsilon^2}\left\{\left(s_1^2+\cdots+s_p^2-s_{p+1}^2-\cdots-s_{p+q}^2\right)-\right.$$

$$\left.-\, s_{p+q}^2\Big) - i\varepsilon\left(s_1^2+\cdots+s_{p+q}^2\right)\right\}^{-\lambda-\frac{n}{2}},$$

or equivalently

$$F[P(t)^{\lambda}] = \frac{M_{\varepsilon,p,q}\,.2^{2\lambda+n}\,\pi^{\frac{n}{2}}\,\Gamma\left(\lambda+\frac{n}{2}\right)}{\Gamma(-\lambda)}\cdot\left(Q-i\varepsilon|s|^2\right)^{-\lambda-\frac{n}{2}}; \qquad (I;5)$$

Where we have put

$$Q = s_1^2+\cdots+s_p^2-s_{p+1}^2-\cdots-s_{p+q}^2;$$

$$|s|^2 = s_1^2+\cdots+s_{p+q}^2;$$

$p+q=n$; n = dimension of the space, and

$$M_{\varepsilon,p,q} = \frac{e^{-\frac{\pi}{4}(p+q)i}}{\left(\sqrt{\varepsilon-i}\right)^p\left(\sqrt{\varepsilon+i}\right)^q\left(1+\varepsilon^2\right)} \qquad (I,6)$$

II. FOURIER TRANSFORM OF GENERALIZED FUNCTIONS DEPENDING ON $\left(P+i\varepsilon|t|^2\right)$.

In this paragraph we shall consider an analogue of the theorem by Trione (cf [2], and [3], p. 23) which allows us to evaluate the Fourier transform of a family of generalized functions depending on the distribution $(P+io)$ as if it depended on $|t|^2$, rotation-invariant distributions, and finally in the result making the substitution of $|s|^2$ by $(Q-io)$.

In the present case we consider the functions $\left(P+i\varepsilon|t|^2\right)$ obtaining the following result. Let $f(z,\lambda)$ be an entire function of the variables *z* and λ, *z* and λ being complex numbers. Let $T\left(P+i\varepsilon|t|^2,\lambda\right)$ be the family of generalized functions of the form

$$T\left(P+i\varepsilon|t|^2,\lambda\right)=\left(P+i\varepsilon|t|^2\right)^{\lambda} f\left(P+i\varepsilon|t|^2\right)^{\lambda} =$$

$$=\left(P+i\varepsilon|t|^2\right)^{\lambda}\sum_{\gamma\geq 0} a_\gamma(\lambda)\left(P+i\varepsilon|t|^2\right)^{\lambda}. \quad \text{(II,1)}$$

Let us consider the family of rotation-invariant generalized functions

$$T\left(|x|^2,\lambda\right)=\left(|x|^2\right)^{\lambda} f\left(|x|^2,\lambda\right)=$$

$$=\left(|x|^2\right)^{\lambda}\sum_{\gamma\geq 0} a_\gamma(\lambda)|x|^{2\gamma}. \quad \text{(II,2)}$$

According to the hypothesis and considerations made by Trione in the theorem about the generalized functions $T(P+io,\lambda)$ and observing that they remain valid for $T\left(P+i\varepsilon|t|^2,\lambda\right)$ the following results is obtained

$$F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=M_{\varepsilon,p,q}F\left[T\left(|x|^2,\lambda\right)\right]_{|\xi|^2\to Q-i\varepsilon|S|^2}. \quad \text{(II;3)}$$

where $M_{\varepsilon,p,q}$ is defined by (I,6) and where the symbol appearing on the right-hand side is the same as in Trione's theorem.

From the above mentioned considerations we have the following

Theorem 1: Let $T\left(P+i\varepsilon|t|^2,\lambda\right)$ be a temperate generalized function, then its Fourier transform is

$$F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=M_{\varepsilon,p,q}.F\left[T|x|^2,\lambda\right]_{|\xi|^2\to Q-i\varepsilon|S|^2}.$$

Furthermore, we have the following

Theorem 2. The same hypothesis as in Theorem 1.

Then

$$F\left[\lim_{\varepsilon\to 0} T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=\lim_{\varepsilon\to 0} F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right].$$

In fact. Let $f(z,\lambda)$ be an entire function of the complex variables *z* and λ:

$$f(z,\lambda)=\sum_{\gamma\geq 0} a_\gamma(\lambda) z^\gamma,$$

and let $T\left(P+i\varepsilon|t|^2,\lambda\right)$ and $T\left(|x|^2,\lambda\right)$ be the families of distributions of the form

$$T\left(P+i\varepsilon|t|^2,\lambda\right)=\left(P+i\varepsilon|t|^2\right)^{\lambda}\sum_{\gamma\geq 0}a_{\gamma}(\lambda)\left(P+i\varepsilon|t|^2\right)^{\gamma}, \qquad \text{(II,4)}$$

and

$$T\left(|x|^2,\lambda\right)=\left(|x|^2\right)^{\gamma}\sum_{\gamma\geq 0}a_{\gamma}(\lambda)|x|^{2\gamma}.$$

Gelfand proved (cf [1], p. 275) that

$$\lim_{\varepsilon\to 0}\left(P+i\varepsilon|t|^2\right)^{\lambda}=(P+io)^{\lambda}.$$

By Theorem 1 we have

$$F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=M_{\varepsilon,p,q}\frac{2^{2\lambda+n}\pi^{\frac{n}{2}}\Gamma\left(\lambda+\frac{n}{2}\right)}{\Gamma(-\lambda)}\cdot\left(Q-i\varepsilon|s|^2\right)^{-\lambda-\frac{n}{2}}$$

or equivalently

$$F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=M_{\varepsilon,p,q}.F\left[T|x|^2,\lambda\right]_{|\xi|^2\to Q-i\varepsilon|S|^2} \qquad \text{(II,5)}$$

Taking the limit en (II,4) when $\varepsilon\to 0$ we have

$$\lim_{\varepsilon\to 0}T\left(P+i\varepsilon|t|^2,\lambda\right)=\lim_{\varepsilon\to 0}\left(P+i\varepsilon|t|^2\right)^{\lambda}\sum_{\gamma\geq 0}a_{\gamma}(\lambda)\left(P+i\varepsilon|t|^2\right)^{\gamma} \qquad \text{(II,6)}$$

If $\mathrm{Re}\,\lambda>-1$ then each term of the second member of (II,6) is a locally integrable function, and therefore in every compact $K\subset R^n$ we have

$$\lim_{\varepsilon\to 0}\left(P+i\varepsilon|t|^2\right)^{\lambda}\sum_{\gamma\geq 0}a_{\gamma}(\lambda)\left(P+i\varepsilon|t|^2\right)^{\gamma}=$$

$$=(P+io)^{\lambda}\sum_{\gamma\geq 0}a_{\gamma}(\lambda)(P+io)^{\gamma}.$$

Then

$$F\left[\lim_{\varepsilon\to 0}T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=F\left[\sum_{\gamma\geq 0}a_{\gamma}(\lambda)(P+io)^{\lambda+\gamma}\right]. \qquad \text{(II,7)}$$

Taking into account (II,5) we get

$$F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=M_{\varepsilon,p,q}\cdot\sum_{\gamma\geq 0}a_{\gamma}(\lambda)\frac{2^{\gamma}\Gamma\left(\lambda+\gamma+\frac{n}{2}\right)}{\Gamma(-\lambda-v)}\cdot\left(Q-i\varepsilon|s|^2\right)^{-\lambda-\frac{n}{2}}$$

and taking the limit when $\varepsilon\to 0$ and by virtue of the uniform convergence over compacts $K\subset R^n$ it turns out that

$$\lim_{\varepsilon\to 0} F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]= e^{-i\frac{\pi}{2}q}\, M_{\varepsilon,p,q}\cdot\sum_{\gamma\geq 0} a_\gamma(\lambda)\frac{2^\gamma\Gamma\left(\lambda+\gamma+\frac{n}{2}\right)}{\Gamma(-\lambda-v)}\cdot\left(Q-i\varepsilon|s|^2\right)^{-\lambda-\frac{n}{2}} \qquad (II;8)$$

By virtue of Trione's Theorem the right-hand members of (II;7) and (II,8) are equal, then

$$F\left[\lim_{\varepsilon\to 0} T\left(P+i\varepsilon|t|^2,\lambda\right)\right]=\lim_{\varepsilon\to 0} F\left[T\left(P+i\varepsilon|t|^2,\lambda\right)\right]. \qquad (II,9)$$

which concludes the theorem.

Aplication

If in (II, 1) we take the limit when $\varepsilon\to 0$, it result

$$\lim_{\varepsilon\to 0}\left(P(t)+i\varepsilon|t|^2\right)=\left(P+i0\right),$$

and from Theorem 2

$$F\left[(P+i0)^\lambda\right]=F\left[\lim_{\varepsilon\to 0}\left(P+i\varepsilon|t|^2\right)^\lambda\right]=\lim_{\varepsilon\to 0}\left\{M_{e,p,q}.2^{2\lambda+n}\pi^{\frac{n}{2}}\frac{\Gamma\left(\lambda+\frac{n}{2}\right)}{\Gamma(-\lambda)}\left(Q-i\varepsilon|t|^2\right)^{-\lambda-\frac{n}{2}}\right\}.$$

And taking into account that:

$$\lim_{\varepsilon\to 0}\left(\sqrt{\varepsilon-i}\right)^p=\left(e^{-\frac{i}{2}\frac{\pi}{2}}\right)^p,$$

$$\lim_{\varepsilon\to 0}\left(\sqrt{\varepsilon+i}\right)^q=\left(e^{+\frac{i}{2}\frac{\pi}{2}}\right)^q,$$

we get

$$F\left[(P+i0)^\lambda\right]=e^{-i\frac{\pi}{2}qi}.2^{2\lambda+n}\frac{\Gamma\left(\lambda+\frac{n}{2}\right)}{\Gamma(-\lambda)}(Q-i0)^{-\lambda-\frac{n}{2}}$$

which coincides with formulae (3), page 284 of [1]

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