ON THE RELATION BETWEEN THE CAUSAL BESSEL DERIVATIVE AND THE MARCEL RIESZ ELLIPTIC AND HYPERBOLIC KERNELS

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RESUMEN: Considerando los núcleos de Riesz como casos particulares del núcleo causal ultrahiperbólico de Bessel se estudia la derivada causal de Bessel de núcleos de Marcel Riez, elíptico o hiperbólico. Utilizando la transformada de Fourier y teniendo en cuenta el producto multiplicativo de las distribuciones $\left(m^2+P\pm i0\right)^{\lambda}$ y de las formalmente análogas $\left(P\pm i0\right)^{\lambda}$ se evaluan dichas derivadas obteniendose resultados que, como casos particulares, contienen a otros debidos a Samko, para el caso del núcleo radial de Riesz, y a Rubin, para el caso del núcleo elíptico de Bessel.

ABSTRACT:: By considering the Riesz kernel as a particular case of the causal Bessel ultrahyperbolic kernel, the relation between the causal Bessel derivative and the Marcel Riesz kernel are presented. We evaluate, via Fourier transform, this relations, taking into account the multiplicative product of certain distributions.

Palabras claves: derivada de Bessel, potencial de Bessel, potencial de Riesz

Key words: Bessel derivative, Bessel potential, Riesz potential

INTRODUCTION

Let R^n be the n dimensional real space and let $t=\left(t_1,t_2,...,t_n\right)$ be a point of it.
Let $P=P(t)$ be a non degenerate quadratic form in n variables in the form

$$P=P(t)=t_1^2+...+t_P^2-t_{P+1}^2-...-t_{p+q}^2, \qquad (I,1)$$
$$p+q=n.$$

The generalized functions $\left(P\pm io\right)^{\lambda}$ were introduced by Gelfand (cf [5]) by the following limit

$$\left(P\pm i0\right)^{\lambda}=\lim_{\varepsilon\to 0}\left(P\pm i\varepsilon|t|^2\right)^{\lambda}, \qquad (I,2)$$

where $\varepsilon>0$, λ a complex number and $|t|^2=t_1^2+\cdots+t_n^2$.

According Gelfand (cf [5], p.287) and from the series development of the Bessel functions, Trione (cf [10]) defined the causal (anticausal) Bessel kernel by the expression:

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$$G_\alpha\left(\mathrm{P}+i0,m,n\right)=C_\alpha\left(m,n\right)\left(\mathrm{P}+i0\right)^{\frac{1}{2}\left(\frac{\alpha-n}{2}\right)}K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(\mathrm{P}+i0)}\right), \qquad (\mathrm{I},3)$$

where $C_\alpha(m,n)=\dfrac{2^{1-\frac{(\alpha+n)}{2}}e^{i\frac{\pi}{2}q}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}$,

m is a positive real number, and $K_\gamma(z)$ is the modified Bessel function of the third kind.

The Fourier transform of a function f is, by definition

$F[f](\xi)=\int_{R^n} e^{i<t,\xi>}f(t)dt$,

where $\langle t,\xi\rangle=\sum_{i=1}^{n}t_i\xi_i$

The Fourier transform of $G_\alpha(\mathrm{P}\pm i0,m,n)$ is (cf [10])

$$\mathrm{F}[\mathrm{G}_\alpha]=\frac{e^{\mathrm{i}\pi\left(\frac{\alpha}{2}\right)}}{(2\pi)^{\frac{n}{2}}}\left(\mathrm{m}^2+Q-i0\right)^{-\frac{\alpha}{2}}, \qquad (\mathrm{I},4)$$

where the distribution $(\mathrm{m}^2+Q\pm i0)^\lambda$ is defined in an analogue maner that in (I, 2) by the following limit.

$$\left(\mathrm{m}^2+Q\pm i0\right)^\lambda=\lim_{\varepsilon\to0}\left(\mathrm{m}^2+Q\pm i\varepsilon|t|\right)^\lambda, \qquad (\mathrm{I},5)$$

were m is a real positive number, (cf[5]).

This distributions are entire distributional function of λ and this is the principal difference with the $(\mathrm{P}\pm i0)^\lambda$ analogues distributions that are analytic in λ except at

$\lambda=-\frac{n}{2}-k,\quad k=0,1,2,...$

Trione (cf [9]) define the ultrahyperbolic Bessel operator of order α of a function φ, or Bessel potential of order α by the convolution

$$B^\alpha\varphi=G_\alpha(\mathrm{P}\pm i0,m,n)*\varphi, \qquad (\mathrm{I},6)$$

where the convolution is taking in the distributional sense.

This Bessel operator $B^\alpha\varphi$ is analogue to the one defined by convolution with the elliptic Bessel kernel introduced by Aronszajn-Smith (cf [1]) and Calderón (cf [2]).

Rubin (cf [6]) inverted these operator by using hypersingular integrals with weighted differences. Taking into account this procedure, to invert the causal Bessel operator, we have defined, in another previous paper, (cf [3]) the “causal hypersingular integral with weighted differences” of a function f as follows

$$\left(\mathrm{T}^\alpha_{l,\varepsilon}\mathrm{f}\right)(x)=\int_{\mathrm{R}^n}\left\{\left(\Delta^l_t f\right)\left(x,\lambda_\alpha\left(\mathrm{k}\left(\mathrm{P}+i0,m,n\right)\right)\right)\right\}\left(\mathrm{P}+i\varepsilon|t|^2\right)^{-\frac{n+\alpha}{2}}dt, \qquad (\mathrm{I},8)$$

where $\alpha \in R,\ \alpha > 0,\ l > \alpha;\ \frac{n+\alpha}{2} \neq 0,-1,-2,...\ \frac{n+\alpha}{2} \neq -\frac{n}{2} - k;\ k = 0,1,2,...$,
using the auxiliary weight function given by

$$\lambda_\alpha(P + i0, m, n) = \frac{2^{1-\frac{n+\alpha}{2}} e^{i\frac{\pi}{2}q} (m^2)^{\frac{1}{2}\left(\frac{n+\alpha}{2}\right)} (P + io)^{\frac{n+\alpha}{4}}}{\Gamma\left(\frac{n+\alpha}{2}\right)} K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2(P + i0)}\right), \qquad \text{(I, 7)}$$

(cf [3]) which is an analogue causal of the weight function due to Rubin (cf [6]).
We have evaluated the Fourier transform of $T^\alpha_{l,e} f$ (cf [3])

$$F\left[T^\alpha_{l,\varepsilon} f\right](\xi) = M_{\varepsilon,p,q}\ d_{n,l}(\alpha)\ F[f](\xi)\left(m^2 + Q - i\varepsilon|\xi|^2\right)^{\frac{\alpha}{2}}, \qquad \text{(I, 9)}$$

where $M_{\varepsilon,p,q} = \frac{e^{-\frac{\pi}{4}(p+q)i}}{(\sqrt{\varepsilon - i})^p (\sqrt{\varepsilon + i})^q (1+\varepsilon)^2}$ and

$$d_{n,l}(\alpha) = \begin{cases} \dfrac{\pi^{\frac{n}{2}+1} A_l(\alpha) e^{i\frac{\pi}{2}q}}{2^\alpha \Gamma\left(1 + \frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)} & \text{if } \alpha \neq 2,4,6,... \\[2ex] \dfrac{(-1)^\alpha \pi^{\frac{n}{2}} e^{i\frac{\alpha}{2}q} 2^{1-\alpha}}{\Gamma\left(1 - \frac{\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)} \dfrac{d}{d\alpha} A_l(\alpha) & \text{if } \alpha = 2,4,6,... \end{cases}$$

(cf [3])
Taking limit for $\varepsilon \to 0$ in (I, 9) we observe that the limit in the right hand member exist and is equal to

$$d_{n,l}(\alpha)\, F[f](\xi)\left(m^2 + Q - i0\right)^{\frac{\alpha}{2}}. \qquad \text{(I, 10)}$$

Then, the limit in the left hand member exists and, by definition, is the operator denoted by $T^\alpha_l f$.
From this result, we have defined the “generalized causal Bessel derivative of order α” D^α by

$$D^\alpha f = \frac{1}{d_{n,l}(\alpha)} T^\alpha_l f \text{ (cf [3])}, \qquad \text{(I, 11)}$$

and from (I, 10) we get

$$F\left[D^\alpha f\right] = \left(m^2 + Q - i0\right)^{\frac{\alpha}{2}} F[f](\xi) \text{ (cf [3])}. \qquad \text{(I,12)}$$

II. THE CAUSAL BESSEL DERIVATIVE OF THE ELLIPTIC BESSEL KERNEL

Let $A\left(|t|^2, m, n\right)$ be the elliptic Bessel kernel introduced by Aronszajn-Smith (cf [1]), Laurent Schwartz (cf [8]) and A.P. Calderón (cf [2]) defined by

$$A\left(|t|^2, m, n\right) = \frac{2^{1-\frac{\alpha+n}{2}}(m^2)^{\frac{n-\alpha}{2}}}{\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\left(|t|^2\right)^{\frac{\alpha-n}{2}} K_{\frac{n-\alpha}{2}}\left(\sqrt{m^2|t|^2}\right); \qquad \text{(II, 1)}$$

Its Fourier transform is (cf [1])

$$F\left[A\left(|t|^2, m, n\right)\right](\xi) = \left(m^2 + |\xi|^2\right)^{-\frac{\alpha}{2}}. \qquad \text{(II, 2)}$$

We observe that the causal Bessel kernel defined in (I, 3) is an analogue causal of the kernel $A\left(|t|^2, m, n\right)$.

By convolution with a function $\varphi \in S$ the elliptic Bessel potential of order α B^{α} is obtained. Moreover, is know that the space S is invariant related to the B^{α} operator, then $B^{\alpha}\varphi$ belonging to S.

From the above considerations we have the following theorems.

Theorem 1

Let $D^{\alpha} f$ be defined as in (I, 11) and let $A\left(|t|^2, m, n\right)$ be the function defined by (II, 1). Then

$$F\left[D^{\alpha} A\left(|t|^2, m, n\right)\right](\xi) = \left(m^2 + Q - i0\right)^{\frac{\alpha}{2}}\left(m^2 + |\xi|^2\right)^{-\frac{\alpha}{2}}, \qquad \text{(II, 3)}$$

in the sense of distributions.

Proof. It result from (I, 12) and (II, 2).

Theorem 2

Let $D^{\alpha} f$ as in (I, 11) and let $B^{\alpha}\varphi$ be the elliptic Bessel potentials of order α. Then

$$F\left[D^{\alpha} B^{\alpha}\varphi\right](\xi) = \left(m^2 + Q - i0\right)^{\frac{\alpha}{2}}\left(m^2 + |\xi|^2\right)^{-\frac{\alpha}{2}} F[\varphi]. \qquad \text{(II,4)}$$

Proof. It result from (I, 12) for $f = B^{\alpha}\varphi;\ \varphi \in S$.

Corollary 1. If in (I, 1) we consider $q = 0$, from (I, 5) and (II, 4) we obtain

$$F\left[D^{\alpha} B^{\alpha}\varphi\right](\xi) = \left(m^2 + |\xi|^2\right)^{\frac{\alpha}{2}}\left(m^2 + |\xi|^2\right)^{-\frac{\alpha}{2}} F[\varphi] = F[\varphi]. \qquad \text{(II,5)}$$

That is the same result that the one due by Rubin (cf [6]) for elliptic Bessel potentials.

The causal Bessel derivative of elliptic Riesz kernel

Taking into account the continuity respect to m of the $\left(m^2+Q\pm i0\right)^{\lambda}$ distribution and that when $m=0, \left(m^2+Q\pm i0\right)^{\lambda}$ reduces to $\left(Q\pm i0\right)^{\lambda}$ and the causal Bessel kernel (I, 3) is the causal Riesz kernel (cf [7]) we have the following.

Theorem 3

Let $R_{\alpha}(\mathrm{P}\pm io,n)$ be the causal Riesz kernel defined as follows. (cf [10]).

$$R_{\alpha}(\mathrm{P}\pm i0,n)=\frac{e^{i\frac{\pi}{2}\alpha}e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n-\alpha}{2}\right)}{2^{\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}(\mathrm{P}\pm i0)^{\frac{\alpha-n}{2}}, \quad \text{(III, 1)}$$

and let $D^{\alpha}f$ defined as in (I, 11). Then

$$F\left[D^{\alpha}R_{\alpha}(\mathrm{P}\pm i0,n)\right]=\left(m^2+Q\mp i0\right)^{\frac{\alpha}{2}}\left(Q\mp i0\right)^{-\frac{\alpha}{2}}. \quad \text{(III, 2)}$$

Proof

Since the Fourier transform of $R_{\alpha}(\mathrm{P}\pm i0,n)$ is (cf[10])

$$F\left[R_{\alpha}(\mathrm{P}\pm i0,n)\right]=\left(Q\mp i0\right)^{-\frac{\alpha}{2}}, \quad \text{(III, 3)}$$

we have (III, 2).

Corollary 1. If we consider $m=0$ in (III, 2) we obtain the causal Riesz derivative of causal Riesz kernel and $F\left[D^{\alpha}R_{\alpha}(\mathrm{P}\pm i0,n)\right]=1$. (III, 4)

Proof: It result from the continuity respect to m of the $\left(m^2+Q\mp i0\right)^{\lambda}$ distribution.

Corollary 2

Taking in (I, 1) $q=0$, from (I, 5) and (II, 7) we have

$$F\left[D^{\alpha}R_{\alpha}\left(|t|^2,n\right)\right]=\left(m^2+|t|^2\right)^{\frac{\alpha}{2}}\left(|t|^2\right)^{-\frac{\alpha}{2}}. \quad \text{(III, 5)}$$

Corollary 3. If we consider $m=0$ and $q=0$ in (III, 2) we have

$$F\left[D^{\alpha}R_{\alpha}\left(|t|^2,n\right)\right]=\left(|t|^2\right)^{\frac{\alpha}{2}}\left(|t|^2\right)^{-\frac{\alpha}{2}}=1. \quad \text{(III, 6)}$$

Which is the same result that the one due by Samko for radial Riesz kernel (cf [7]).

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